Tantrices of Spherical Curves

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1. INTRODUCTION

(1.1) If the speed of a smooth closed curve σ on the unit sphere $\mathbf{S}^2$ never vanishes, than σ's *normalized* velocity vector $\tau := \dot{\sigma}/|\dot{\sigma}|$ sweeps out a new closed curve on $\mathbf{S}^2$, often called the *tangent indicatrix* of σ. Here we shall simply call τ the *tantrix* of σ.

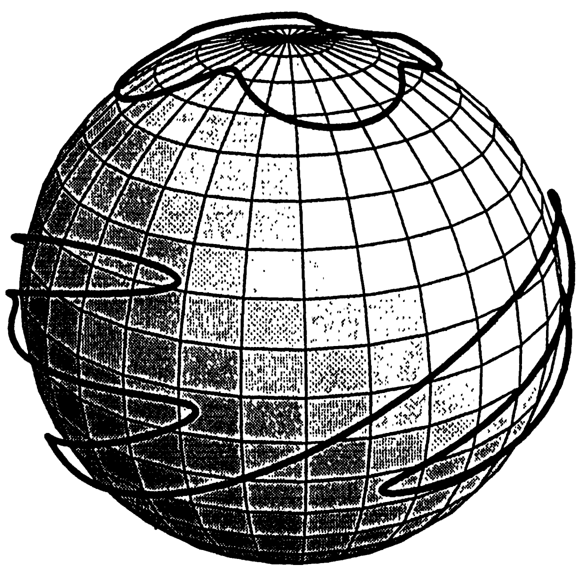

Figure 1. A closed curve σ on $\mathbf{S}^2$ (shorter loop near north pole) and its tantrix.

While every loop σ on $\mathbf{S}^2$ with non-vanishing speed defines a "tantricial" loop in this way, *not every loop is tantricial.* Here, we will completely expose the non-obvious—but lovely—obstruction to this converse.

To begin, note that *if the speed of σ never vanishes, neither will that of τ.* In fact, the speed of τ, computed relative to arclength along σ, gives the *curvature* of σ as a curve in $\mathbf{R}^3$. But σ, lying on $\mathbf{S}^2$, can nowhere approximate a straight line to second order: its curvature—the speed of τ—will never vanish. Referring non-experts to the sidebar discussions of "immersion" and "arclength" for further details, we conclude more precisely that *if σ immerses the circle in $\mathbf{S}^2$, so will τ.*

This observation provokes our main question:
When does one immersed circle on $\mathbf{S}^2$ form the tantrix of another?

Immersion. Roughly speaking, a curve in $\mathbf{S}^2$ constitutes an *immersion* if it has no corners or cusps, though it may cross itself. Analytically, one guarantees these properties by insisting that *the curve's velocity never-vanishes.* A mapping $\sigma\colon \mathbf{S}^1 \to \mathbf{S}^2$ thus forms an *immersion of the circle into the sphere* iff for all $\theta \in \mathbf{R}$, we have

$$\left|\frac{d}{d\theta}\sigma(e^{i\theta})\right| > 0$$

In this article, we shall deal only with twice continuously differentiable ("smooth") immersions.

Arclength Parametrization. A basic fact from the differential geometry of curves states that one can reparametrize any immersion using *arclength.* This means we can give the curve *unit-speed* relative to some new parameter s. In the context above, for example, we can arrange

$$\left|\frac{d}{ds}\sigma(e^{i\theta(s)})\right| \equiv 1$$

The tantrix τ of σ will now *equal* σ's velocity vector: $\tau = d\sigma/ds$. As claimed in our introduction, τ will then *immerse* the circle into $\mathbf{S}^2$—$d\tau/ds$ will never vanish. In fact, we can easily prove $|d\tau/ds| \geq 1$:

$$0 = \frac{1}{2}\frac{d^2}{ds^2}(|\sigma|^2) = \left|\frac{d\sigma}{ds}\right|^2 + \frac{d^2\sigma}{ds^2}\cdot\sigma = 1 + \frac{d\tau}{ds}\cdot\sigma,$$

hence (by Cauchy-Schwartz)

$$(*) \qquad \left|\frac{d\tau}{ds}\right| = \left|\frac{d\tau}{ds}\right||\sigma| \geq \left|\frac{d\tau}{ds}\cdot\sigma\right| = 1.$$

Since $|d^2\sigma/ds^2|$ gives the *curvature* of a unit-speed curve σ, and we have $d\tau/ds = d^2\sigma/ds^2$, this shows too that *any curve on $\mathbf{S}^2$ has curvature at least 1.*

Joel Weiner recently discovered the answer "by accident" while working on the seemingly unrelated topic of flat tori in $\mathbf{S}^3$:

(1.2) Theorem. ([W]) *An immersed circle τ in $\mathbf{S}^2$ forms a tantrix if and only if it has total geodesic curvature zero, and contains no subarc with total geodesic curvature π.*

While Weiner derives this simple fact as a corollary to his results on flat tori, he remarks that "it would be nice to have a curve-theoretic proof" of the result. Here we provide such a proof, and present some related facts. For one, we have

(1.3) Theorem. *An immersed circle in $\mathbf{S}^2$ and its tantrix share a regular homotopy class. A tantrix in the equator's class always bounds oriented area 2π (mod 4π). A tantrix in the other class bounds area zero.*

As we discuss in a sidebar (§5), *regular homotopy* means "homotopy through immersions," and in $\mathbf{S}^2$, immersed circles come in just two regular homotopy flavors, represented by single and double traversals of the equator respectively. Notice that these two curves each form their own tantrices, and indeed, bound areas 2π and $0 \pmod{4\pi}$ respectively. (See §5 for the notion of "oriented area $\bmod 4\pi$.")

The claim about area in this Theorem suggests a connection with the classical result known as "Jacobi's Theorem for space curves," which appears in standard texts such as Chern [**Ch**, p. 44], Spivak [**Sp**, §6.12], and DoCarmo [**DoC**]. In fact, Jacobi's theorem follows immediately from the earlier half of Weiner's theorem, which we can state like this:

Proposition 3.1. *The tantrix of an immersed circle in* $\mathbf{S}^2$ *always has total geodesic curvature zero, and if non-self-intersecting, bounds area* 2π.

We postpone the elementary proof of this fact to §3, but we show right now how to deduce Jacobi's Theorem from it:

(1.4) Corollary. ("Jacobi's Theorem") *The principal normal indicatrix of a closed space curve with non-vanishing curvature, if embedded,* bisects *the area of* $\mathbf{S}^2$.

(1.5) *Proof.* The unit tangent vector T along a closed space curve γ maps $\mathbf{S}^1$ into $\mathbf{S}^2$, and here, *immerses* $\mathbf{S}^1$, since the curvature κ of γ never vanishes. Indeed, when s represents arclength along γ, the Frenet formula $dT/ds = \kappa N$, shows that the tantrix of T coincides with the *normal* indicatrix N of γ. If N never crosses itself, it bounds a domain $\Omega \subset \mathbf{S}^2$, with area $0 < |\Omega| < 4\pi$. Proposition 3.1 then forces $|\Omega| = 2\pi = \frac{1}{2}|\mathbf{S}^2|$. Q.E.D.

Note that Jacobi's Theorem does *not*, conversely, imply Proposition 3.1. For, the typical closed curve on $\mathbf{S}^2$ does *not* form the tangent indicatrix of a closed space curve; the latter always have length at least 2π, (see [**Ch**, §4], for instance), and can never lie in an open hemisphere (exercise). *We therefore submit that Proposition 3.1—not Jacobi's theorem—would better serve texts like* [**Ch**], [**Sp**], *and* [**DoC**]. With the aim of emphasizing this possibility, we prove the "if and only if" statement of Theorem 1.2 as separate subsidiary results—Propositions 3.1 and 4.1—and prove them by completely elementary techniques.

To deal with the subtler notions of oriented area and regular homotopy invoked by Theorem 1.3, we must resort to more sophisticated means in §5. We have tried to write §5 so that novice readers can follow its main points, if not its details, however, so the earlier sections borrow nothing from this final one.

Incidentally, Theorem 1.3 implies an amusing corollary:

(1.6) Corollary. *Any immersed circle regularly homotopic to a figure-eight in* $\mathbf{S}^2$ *must cross some great circle orthogonally—and in the same direction—at least twice.*

Proof: One can regularly homotope any figure-eight in $\mathbf{S}^2$ to a double-equator, and the *tantrix* of such a curve must then self-intersect, since the Proposition now excludes it from the regular homotopy class of the single-equator. A self-crossing of the tantrix, however, signals two or more points on the original closed curve with the same oriented tangent vector. The corollary follows directly, because the set of points on $\mathbf{S}^2$ that share a particular tangent direction form the great circle perpendicular to that direction. Q.E.D.

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2. CURVES ON $\mathbf{S}^2$ AND A KEY LEMMA. Here we prepare for later arguments with some elementary calculations for curves on $\mathbf{S}^2$, and prove an easy—but very illuminating—lemma.

(2.1) Consider an arbitrary immersed curve on $\mathbf{S}^2$, and call it τ, since we will soon want to ask whether it arises as the tantrix of some other spherical curve σ. Refer τ to an arclength parameter t (we will save "s" for arclength along σ), so that $\dot{\tau} := d\tau/dt$ has length 1. Define also the unit normal ν to τ in $\mathbf{S}^2$ via $\nu := \tau \times \dot{\tau}$, so that $\{\tau, \dot{\tau}, \nu\}$ forms an oriented orthonormal basis for $\mathbf{R}^3$, with $\dot{\tau}$ and ν spanning the tangent plane to $\mathbf{S}^2$ at $\tau(t)$ for each t. By definition, when one expands the *second* derivative $\ddot{\tau}$ in terms of this frame, the coefficient of ν gives the *geodesic curvature* κ_g of τ. To get the two remaining coefficients, we simply compute

$$\ddot{\tau} \cdot \dot{\tau} = \frac{1}{2}\frac{d}{dt}|\dot{\tau}|^2 = 0,$$

Geodesic Curvature. As mentioned above, the acceleration vector of a unit-speed space curve gives (via its length) the *curvature* κ of that curve. One can parametrize any immersed curve by arclength, making this definition quite general. (*Exercise:* For a radius-r circle in the plane, $\kappa = 1/r$.)

For unit-speed curves on *surfaces*, we get the *geodesic* curvature—κ_g—by measuring the length of the *surface-tangential component* of the acceleration. (*Exercise*: For a radius-r circle on $\mathbf{S}^2$, $\kappa_g = \sqrt{1 - r^2}/r$.) As shown in §3, geodesic curvature relates very closely to parallelism. Indeed, by Equation 3.2.1, κ_g gives the rate at which a *parallel* vectorfield rotates relative to the *tangen* vectorfield of a unit-speed curve. (*Exercise*: How much will a Foucault pendulum at latitude ϕ precess in 24 hours?)

Geodesics—curves with $\kappa_g \equiv 0$—play a major role in Differential Geometry because they provide the shortest paths connecting pairs of points on surfaces (and on higher dimensional "surfaces").

and

$$\ddot{\tau} \cdot \tau = \frac{1}{2}\frac{d^2}{dt^2}\left(|\tau|^2\right) - |\dot{\tau}|^2 \equiv -1.$$

We thus have

$$\ddot{\tau} = \kappa_g \nu - \tau. \tag{2.1.1}$$

Among other things, this fact implies the spherical "Frenet" formula

$$\begin{aligned}\dot{\nu} &= \frac{d}{dt}(\tau \times \dot{\tau}) = \dot{\tau} \times \dot{\tau} + \tau \times \ddot{\tau} = \tau \times (\kappa_g \nu - \tau) = \kappa_g \tau \times \nu \\ &= -\kappa_g \dot{\tau}\end{aligned} \tag{2.1.2}$$

Next, consider a unit vector field σ tangent to $\mathbf{S}^2$ along τ. Characterize the angle

ϕ between σ and $-\dot{\tau}$ at each time t by the equation

$$\sigma(t) = -\cos \phi(t) \cdot \dot{\tau}(t) + \sin \phi(t) \cdot \nu(t). \tag{2.1.3}$$

Differentiate this equation, and expand in terms of $\dot{\tau}$, ν, and τ, using Equations 2.1.1 and 2.1.2:

$$\begin{aligned}\dot{\sigma} &= \dot{\phi} \sin \phi \dot{\tau} - \cos \phi \ddot{\tau} + \dot{\phi} \cos \phi \nu + \sin \phi \dot{\nu} \\ &= \sin \phi\left(\dot{\phi} - \kappa_g\right)\dot{\tau} + \cos \phi\left(\dot{\phi} - \kappa_g\right)\nu + \cos \phi \tau \\ &= \left(\dot{\phi} - \kappa_g\right)(\sin \phi \dot{\tau} + \cos \phi \nu) + \cos \phi \tau \end{aligned} \tag{2.1.4}$$

We will soon apply these facts, but first we record a lemma. Though very simple, this lemma has a surprise bottom line invoking the key notion of *parallelism* for vectorfields tangent to $\mathbf{S}^2$ along a curve (see sidebar).

Parallel Vectorfields along Curves on $\mathbf{S}^2$. In the *plane*, we call a vectorfield $\mathbf{v}$ *parallel* along a path γ if it has constant length and direction:

$$\mathbf{v} \text{ parallel} \quad \Leftrightarrow \quad \frac{d}{dt}\mathbf{v}(\gamma(t)) \equiv 0 \text{ (plane)}$$

Along a curve in $\mathbf{S}^2$, however, one can't generally find a vectorfield *tangent* to the sphere with this property. So we can't define parallelism by the complete vanishing of a spherical vectorfield's derivative. One can, however, ask the derivative to vanish *tangentially*, thus defining $\mathbf{v}$ as *parallel* along γ in $\mathbf{S}^2$ if its derivative always points *normally* to $\mathbf{S}^2$ along γ. Since each point in $\mathbf{S}^2$ *equals* the normal to $\mathbf{S}^2$ at that point, we then have

$$\mathbf{v} \text{ parallel} \quad \Leftrightarrow \quad \frac{d}{dt}\mathbf{v}(\gamma(t)) = f(t)\gamma(t) \text{ (sphere).}$$

for some scalar function f.

This notion of parallism has many applications in Geometry, and even in Physics. For example, as it repeatedly sweeps through its lowest point, the velocity vector of a Foucault Pendulum traces out a parallel vectorfield along a circle of latitude on the rotating Earth—showing physically that the parallel extension of an initial starting vector along a loop in $\mathbf{S}^2$ (*e.g.*, a circle of latitude in the "Foucault" case) won't generally end up where it started after traversing the loop!

(2.2) Lemma. *If s and t denote oriented arclength parameters along a curve σ immersed in $\mathbf{S}^2$, and along its tantrix τ respectively, then*

$$\dot{\sigma} := \frac{d\sigma}{dt} = \tau\frac{ds}{dt} \quad \textit{and} \quad \frac{ds}{dt} > 0.$$

One can therefore regard any immersed curve σ in $\mathbf{S}^2$ as a parallel *vectorfield tangent to $\mathbf{S}^2$ along its own tantrix.*

Proof: Write t as a function of s by basing an arclength integral at some value s_0 of s:

$$t(S) = \int_{s_0}^{S}\left|\frac{d\tau}{ds}\right| ds.$$

We thus have $t'(s) = |d\tau/ds|$—the speed of τ— which must exceed 1, as discussed in our paper's second paragraph (see also Equation $(*)$ of the sidebar item on *Arclength*). In particular, $t'(s) > 0$. Furthermore, $d\tau/ds = d^2\sigma/ds^2$, so our standing smoothness assumption on σ ensures continuity of t'. The Inverse Function Theorem now makes s a function of t, with

$$\frac{ds}{dt} = \frac{1}{t'(s)} > 0,$$

and

$$\dot{\sigma} = \frac{d\sigma}{ds}\frac{ds}{dt} = \tau\frac{ds}{dt}.$$

Having derived the asserted formulas, we now simply observe that $\tau(s)$ coincides with the normal to $\mathbf{S}^2$ at $\tau(s)$, and $\sigma(s)\cdot\tau(s) = d/ds(\frac{1}{2}|\sigma|^2) \equiv 0$, so $\sigma(s)$ lies *tangent* to $\mathbf{S}^2$ at $\tau(s)$ for each s. Moreover, our expression for $\dot{\sigma}$ leaves it with *no* component tangent to $\mathbf{S}^2$. This makes σ *parallel* along τ. Q.E.D.

3. AN ELEMENTARY ANTECEDENT TO JACOBI'S THEOREM. Before indulging in a more complete analysis of tantrices, we supply the easy Proposition that implies Jacobi's Theorem via the argument given in §1.5. Recall that the integral of κ_g over an entire curve defines that curve's *total geodesic curvature.*

(3.1) Proposition. *The tantrix of an immersed circle in* $\mathbf{S}^2$ *always has total geodesic curvature zero, and if embedded, bounds area* 2π.

(3.2) *Proof.* Let τ denote the tantrix of an immersed circle $\sigma \subset \mathbf{S}^2$. By Lemma 2.2, we may regard σ as a vectorfield tangent to $\mathbf{S}^2$ along τ, so Equation 2.1.4 applies. Comparing the expression given there for $\dot{\sigma}$ with the one given by Lemma 2.2, we immediately deduce

$$\dot{\phi} = \kappa_g. \tag{3.2.1}$$

Moreover, σ closes up, making ϕ L-periodic, L denoting the length of the tantrix. Hence

$$0 = \phi(L) - \phi(0) = \int_0^L \dot{\phi}\, dt = \int_\tau \kappa_g.$$

This proves the Proposition's first claim, and Gauss-Bonnet does the rest:

$$|\Omega| = \int_\Omega 1\, dA = \int_\Omega K dA = 2\pi - \int_\tau \kappa_g = 2\pi. \qquad \text{Q.E.D.}$$

4. INVERTING A TANTRIX. Implicitly, Lemma 2.2 suggests an elegant way to "invert" a tantrix—*i.e.*, to find an immersed circle $\sigma \subset \mathbf{S}^2$ having a given curve τ as tantrix. Namely, *one should simply parallel-translate a unit vector tangent to* $\mathbf{S}^2$ *around* τ. But even when the total geodesic curvature of τ vanishes—an obvious necessity by Proposition 3.1—this procedure can fail: it may not produce an immersion. For instance, along the equator, the parallel "vertical" vectorfield $(0, 0, 1)$ traces out only one point—hardly an immersed circle. On the other hand, parallel-translating any *non*-vertical unit tangent around the equator yields a circle of latitude, whose tantrix *does* give back the equator. Proposition 4.1 below will resolve this paradox using the oscillation of a total curvature function.

To define that function, observe that the vanishing of total geodesic curvature on an immersed circle $\tau \subset \mathbf{S}^2$

$$\int_\tau \kappa_g = 0,$$

ensures that κ_g has a continuous antiderivative—call it ϕ_τ—on τ. Geometrically, ϕ_τ measures the angle between $-\dot\tau$ and some parallel unit vectorfield along τ; this follows from Equations 2.1.3 and 2.1.4. Of course, τ boasts an entire circle of such vectorfields; we have specified ϕ_τ only up to addition of a constant. But no such ambiguity afflicts the *oscillation* of ϕ_τ, defined via

$$\operatorname{osc} \phi_\tau := \sup \phi_\tau - \inf \phi_\tau .$$

Moreover, Equation 3.2.1 says $\dot\phi_\tau = \kappa_g$. So $\operatorname{osc}\phi_\tau$ *measure the maximum total geodesic curvature of any subarc of* τ.

(4.1) Proposition. *On the tantrix τ of an immersed circle σ in $\mathbf{S}^2$, we always have $\operatorname{osc}\phi_\tau < \pi$; i.e., no subarc of τ has total geodesic curvature π. Conversely, any immersed circle τ in $\mathbf{S}^2$ having total geodesic curvature zero and $\operatorname{osc}\phi_\tau < \pi$, forms the tantrix of some other immersed circle σ in $\mathbf{S}^2$.*

Proof: Recall that by Lemma 2.2, σ lies tangent to $\mathbf{S}^2$ along τ, with

$$\dot\sigma = \tau \frac{ds}{dt} \quad \text{and} \quad \frac{ds}{dt} > 0.$$

These two facts produce immediate consequences via Equation 2.1.4. Namely, on denoting by ϕ the angle between σ and $-\dot\tau$ (as characterized by Equation 2.1.3), 2.1.4 implies

$$\dot\phi = \kappa_g \quad \text{and} \quad \cos\phi = \frac{ds}{dt} > 0.$$

The first identity here shows that ϕ antidifferentiates κ_g; we can take $\phi_\tau = \phi$. The second then forces $-\pi/2 < \phi_\tau < \pi/2$, which clearly means $\operatorname{osc}\phi_\tau < \pi$, and proves the Proposition's first statement.

To get the converse, consider an immersed circle τ in $\mathbf{S}^2$ with total geodesic curvature zero—so that κ_g has a continuous antiderivative—and assume $\operatorname{osc}\phi_\tau < \pi$. This latter restriction clearly lets us choose an antiderivative ϕ_τ for κ_g with $-\pi/2 < \phi_\tau < \pi/2$.

Having done so, let t denote arclength along τ, and construct the unit normal $\nu = \dot\tau \times \tau$ along τ as in §2 above. If we now *define* a tangent vectorfield σ along τ using Equation 2.1.3 with ϕ replaced by ϕ_τ, so that

$$\sigma(t) = -\cos\phi_\tau(t)\dot\tau(t) + \sin\phi_\tau(t)\nu(t),$$

then Equation 2.1.4 immediately forces $\dot\sigma = \cos\phi_\tau \tau$, because $\dot\phi_\tau - \kappa_g = 0$. Since $\cos\phi_\tau$ never vanishes, this makes σ an immersion, with tantrix τ. Q.E.D.

5. AREAS AND REGULAR HOMOTOPY. We now turn to Theorem 1.3, restated here for the reader's convenience:

Theorem 1.3. *An immersed circle in $\mathbf{S}^2$ and its tantrix share a regular homotopy class. A tantrix in the equator's class always bounds area 2π. A tantrix in the other class bounds area zero.*

Regular Homotopy of Curves. *Homotopy* means "continuous deformation." For instance, if we deform a round circle in the plane into a long, narrow ellipse by stretching along an axis, we create a homotopy joining these two closed curves. A homotopy qualifies as *regular* when the curve begins and remains *immersed* throughout the deformation.

Two curves belong to the same regular homotopy *class* if some regular homotopy joins them—like the circle and ellipse mentioned above. Every immersed circle in the *plane* homotopes regularly to an n-times traversed circle for some unique integer n (add $+1$ for each counterclockwise traversal, -1 for clockwise traversals), or to a figure-eight ($n = 0$). In particular, such curves form infinitely many *different* classes. For instance, a simple clockwise circle won't deform into a counterclockwise one without passing through a "cusp" at some intermediate step:

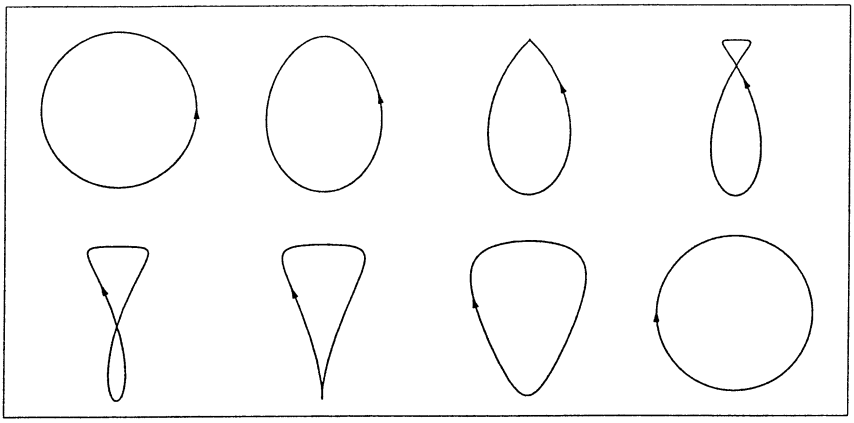

Figure 2. A typical—hence *non*-regular—homotopy from a $+1$ circle to a -1 circle. Like this one, any homotopy between counterclockwise and clockwise circles must pass through a cusp at some stage.

On the *sphere*, however, the situation differs dramatically. For instance, one can homotopically reverse the orientation of a longitudinal circle by simply rotating it 180°, keeping its north and south poles fixed. Moreover, by generalizing this "trick," one can show that on $\mathbf{S}^2$, *immersed circles form only* **two** *regular homotopy classes*. The equator, traced once (or any *odd* number of times) represents one class. Traced twice (or any even number of times), it represents the other class. (*Exercise:* Convince yourself that on $\mathbf{S}^2$, figure-eights belong to the latter class.) For further elaboration of this fascinating topic, see **[P]**.

(5.1) *Areas.* Since the tantrix of an immersed circle can cross itself, thereby cutting the sphere into many sub-domains, the *area* it bounds requires careful definition. One *can* make a good definition, however, because a closed, immersed curve in $\mathbf{S}^2$ always bounds a 2-dimensional *homology class*, and one may assign areas to such classes very naturally.

To see how, let C denote any union of oriented, immersed circles in $\mathbf{S}^2$. Suppose we have a smooth surface S with boundary, and a smooth mapping p: $S \to \mathbf{S}^2$ such that $p(\partial S) = C$. Given C, standard differential topology guarantees the existence of an S and a p with these attributes: in fact, one can extend any immersion with image C smoothly (through not generally as an immersion) to any smooth surface that spans its domain. Now let ω denote the standard area 2-form on $\mathbf{S}^2$. The prescription

$$\operatorname{area}(C) := \int_S p^*\omega \pmod{4\pi} \tag{5.1.1}$$

makes good sense; it depends on neither S, nor p, since it returns $4\pi \deg(p)$ when $\partial(S) = 0$ (*e.g.*, by the "degree formula" in **[G & P]**). Equation 5.1.1 clearly corroborates all familiar definitions of area for closed curves which *don't* self-intersect in $\mathbf{S}^2$—just choose the subdomain of $\mathbf{S}^2$ bounded by C for S. So we define the area bounded by general closed *immersed* curves using this prescription.

(5.2) *The Unit Tangent Bundle of* $\mathbf{S}^2$. A spanning surface S arises naturally in our proof of Theorem 1.3; it will lie in the 3-dimensional space of all unit vectors tangent to $\mathbf{S}^2$—the sphere's *unit tangent bundle* US^2. We may specify such vectors using their base-points in $\mathbf{S}^2$ and their unit direction vectors—also points in $\mathbf{S}^2$. So we shall regard $\mathbf{US}^2$ as the following subset of $\mathbf{S}^2 \times \mathbf{S}^2$ (in $\mathbf{R}^3 \times \mathbf{R}^3$):

$$\mathbf{US}^2 := \{(\mathbf{u}, \mathbf{v}) \in \mathbf{S}^2 \times \mathbf{S}^2 : \mathbf{u} \cdot \mathbf{v} = 0\}.$$

At each base point $\mathbf{u} \in \mathbf{S}^2$, we find a circle of unit tangent vectors $\mathbf{v}_u$. This makes $\mathbf{US}^2$ a bundle of circles over the 2-sphere. Let p denote the bundle projection that sends each circular fiber to its base-point:

$$p : \mathbf{US}^2 \to \mathbf{S}^2, \quad p(\mathbf{v}_u) := \mathbf{u}.$$

The formula

$$(\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{u}, \cos\theta\mathbf{v} + \sin\theta(\mathbf{u} \times \mathbf{v}))$$

rotates all fibers through angle θ (counterclockwise, as viewed from outside the sphere), so by applying $d/d\theta|_{\theta=0}$, we harvest a smooth unit vectorfield on $\mathbf{US}^2$ which flows tangent to the fibers; namely, $(0, \mathbf{u} \times \mathbf{v})$ at the point $(\mathbf{u}, \mathbf{v})$.

Call the one-form dual to this vectorfield α, so that $\alpha(\mathbf{x}) := (0, \mathbf{u} \times \mathbf{v}) \cdot \mathbf{x}$ for any $\mathbf{x}$ tangent to $\mathbf{US}^2$ at $(\mathbf{u}, \mathbf{v})$.† The exterior derivative $d\alpha$, it turns out, gives the *curvature 2-form* on $\mathbf{US}^2$. So since $\mathbf{S}^2$ has constant curvature 1, $d\alpha$ simply pulls back 1 times the area form on $\mathbf{S}^2$, via the projection p:

$$d\alpha = p^*\omega. \tag{5.2.1}$$

Readers unfamiliar with this equation may compute $d\alpha$ directly using the fact that

$$d\alpha(X,Y) = (\nabla_X \alpha)(Y) - (\nabla_Y \alpha)(X) = D_X(0, u \times v) \cdot Y - D_Y(0, u \times v) \cdot X$$

whenever X and Y are tangent to $\mathbf{US}^2$.

Our main argument now comes easily:

(5.3) *Proof of Theorem* 1.3. The tantrix of an immersed circle again *immerses* the circle, as we have noted several times. Note also that multiples of the equator reparametrize their own tantrices. So if we deform a curve σ in $\mathbf{S}^2$ through immersions to some multiple of the equator, its tantrix flows simultaneously to that

†The knowledgeable reader may recognize α as the standard *connection* form on $\mathbf{US}^2$. In particular, if one lifts a smooth unit-speed curve $\tau(t)$ on $\mathbf{S}^2$ to the curve $\gamma = (\tau, \dot\tau)$ in $\mathbf{US}^2$, then $\alpha(\dot\gamma(t))$ returns the geodesic curvature of τ at time t. We shall evaluate α along a *different* lift to prove Theorem 1.3, however.

same multiple equator—and likewise through immersions. In particular, any immersed circle shares a regular homotopy class with its tantrix, as the Theorem's first statement claims.

To get the statements about area, let τ denote the tantrix of an immersed circle σ in $\mathbf{S}^2$, and refer both to an arclength parameter t along τ. Lift τ to the curve $\Lambda_\tau = (\tau, \sigma)$ in $\mathbf{US}^2$. The collinearity of $\dot{\sigma} := d\sigma/dt$ with τ (Lemma 2.2) then makes Λ_τ everywhere *horizontal*—orthogonal to the fibers—in $\mathbf{US}^2$, as an easy calculation shows:

$$\alpha(\dot{\Lambda}_\tau) = (0, \tau \times \sigma) \cdot (\dot{\tau}, \dot{\sigma}) = (\tau \times \sigma) \cdot \dot{\sigma} \equiv 0. \tag{5.3.1}$$

Now consider some multiple of the equator—call it e: $\mathbf{S}^1 \to \mathbf{S}^2$—and suppose we can deform σ to e through immersions. By lifting the tantrix of each curve in the resulting regular homotopy to $\mathbf{US}^2$ as just described, we smoothly map an annulus S into $\mathbf{US}^2$, with $\partial(p(S)) = p(\Lambda_\tau - \Lambda_e) = \tau - e$. We can therefore calculate the area that $\tau - e$ bounds in $\mathbf{S}^2$ by applying Stokes' Theorem in conjunction with our area prescription (5.1.1), the formula for $d\alpha$ (5.2.1), and the horizontality of lifted tantrices (5.3.1):

$$\operatorname{area}(\tau - e) = \int_S p^*\omega = \int_S d\alpha = \int_{\partial S} \alpha = \int_{\Lambda_\tau - \Lambda_e} 0 = 0.$$

It follows immediately that τ and e bound the same area in $\mathbf{S}^2$. Since the Theorem's claims about area clearly hold for multiples of the equator, hence for e, we now get the same for the arbitrary tantrix τ. Q.E.D.

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